Characterizing Climate Variability in Models and Observations via the Spectral Theory of Dynamical Systems

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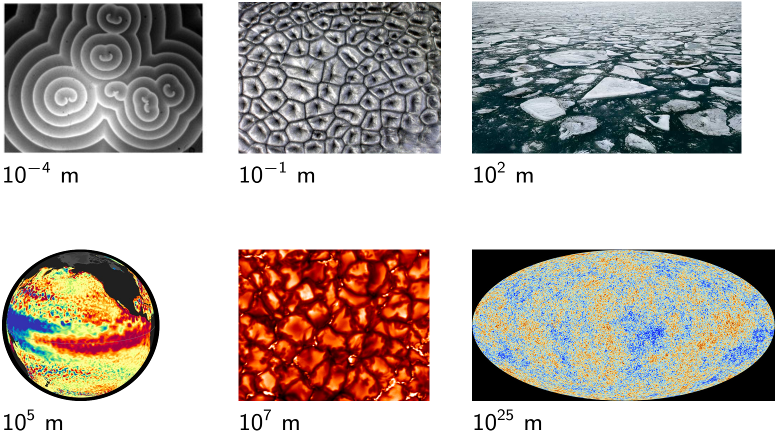
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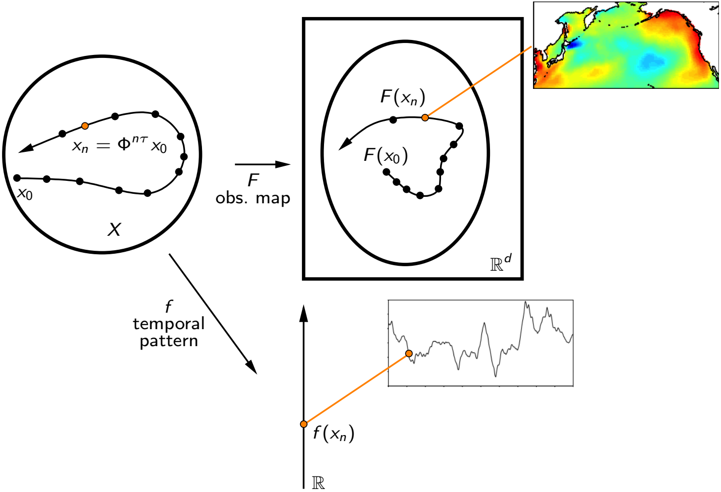

$x_n = \Phi^{n\tau} x_0$
x_0
X
F
obs. map
$F(x_n)$
$F(x_0)$
$\mathbb{R}^d$
f
temporal
pattern
$f(x_n)$
$\mathbb{R}$

HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

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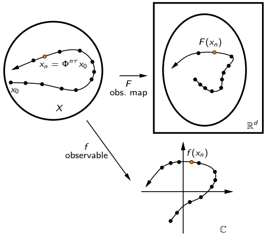
In recent years the theory of Hilbert space and its linear transformations has come into prominence.[1] It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions—linear differential or integral equations and the like, in those relating to harmonic analysis, and in many phenomena due to the operation of the laws of chance, the essential rôle is played by certain linear transformations in Hilbert space. And the importance of the theory in quantum mechanics is known to all. It is the object of this note to outline certain investigations of our own in which the domain of this theory has been extended in such a way as to include classical Hamiltonian mechanics, or, more generally, systems defining a steady n-dimensional flow of a fluid of positive density.

Spaces of observables



- An **observable** is a complex-valued function f on the state space X:

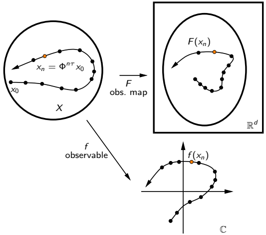
$$\mathcal{F} = \{f : X \mapsto \mathbb{C}\}$$

- $\mathcal{F}$ is a **linear space** even if X is not

$$h = f + g \iff h(x) = f(x) + g(x), \quad f, g, h \in \mathcal{F}$$
$$g = \sigma f \iff g(x) = \sigma f(x), \quad f, g \in \mathcal{F}, \quad \sigma \in \mathbb{C}$$

- $\mathcal{F}$ is **infinite-dimensional** even if X is not

Operator-theoretic representation of dynamical systems



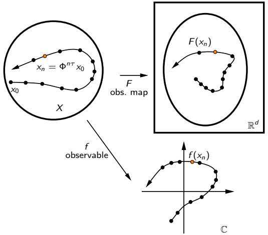
- Define the **Koopman operator** $U^t : \mathcal{F} \mapsto \mathcal{F}$, $t \in \mathbb{R}$, by

$$U^t f(x) = f(\Phi^t x), \quad x \in X$$

- U^t is a **linear operator**, even if Φ^t is nonlinear

$$U^t(f + g) = U^t f + U^t g, \quad f, g \in \mathcal{F}$$
$$U^t(\sigma f) = \sigma U^t f, \quad f \in \mathcal{F}, \quad \sigma \in \mathbb{C}$$

Operator-theoretic representation of dynamical systems



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Without approximation, it is possible to represent a finite-dimensional nonlinear dynamical system by an infinite-dimensional system of linear operators acting on observables

Spectral properties of Koopman operators

- Given an **ergodic probability measure** μ of the dynamical system, we restrict attention to the **Hilbert space**

$$H = \left\{ f \in \mathcal{F} : \int_X |f|^2 \, d\mu < \infty \right\}, \quad \langle f, g \rangle = \int_X f^* g \, d\mu$$

- On H, U^t is a **unitary operator**, $U^{t*} U^t = U^t U^{t*} = I$
- Its eigenvalues and eigenfunctions, $U^t z_k = \Lambda_k z_k$, have the properties

$$\Lambda_k = e^{\mathrm{i}\omega_k t}, \quad \langle z_j, z_k \rangle = \delta_{jk}, \quad U^t(z_j z_k) = e^{\mathrm{i}(\omega_j + \omega_k)t} z_j z_k$$

- The eigenfunctions z_k evolve as **periodic observables** at frequencies intrinsic to the dynamical system,

$$U^t z_k(x) = e^{\mathrm{i}\omega_k t} z_k(x)$$

- Koopman eigenfunctions induce an **invariant splitting**

$$H = \mathcal{D} \oplus \mathcal{D}^\perp, \quad \mathcal{D} = \overline{\operatorname{span}\{z_k\}}$$

- Observables in $\mathcal{D}$ have a **quasiperiodic** evolution

$$f = \sum_k c_k z_k \in \mathcal{D} \quad \implies \quad U^t f = \sum_k e^{\mathrm{i}\omega_k t} c_k z_k$$

- Observables in $\mathcal{D}^\perp$ exhibit a **mixing behavior**

$$f \in \mathcal{D}^\perp \quad \implies \quad \frac{1}{T} \int_0^T |\langle g, U^t f \rangle| \, dt \xrightarrow[T \to \infty]{} 0, \ \forall g \in H$$

Kernel methods for Koopman eigenfunction approximation

The eigenvalue problem for U^t can be formulated equivalently in terms of the **generator** V,

$$Vf = \lim_{t \to 0} \frac{U^t f - f}{t}, \quad V^* = -V, \quad U^t = e^{tV}$$

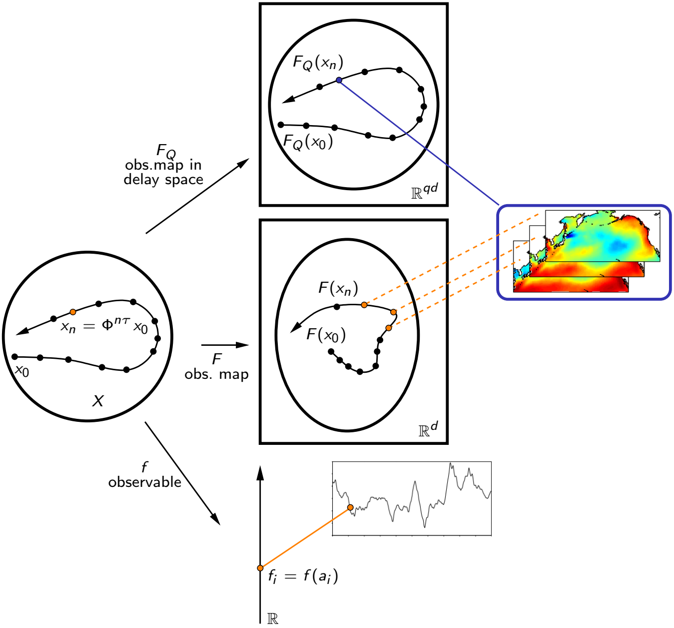
$$U^t z_k = e^{\mathrm{i}\omega_k t} z_k \quad \Longleftrightarrow \quad V z_k = \mathrm{i}\omega_k z_k$$

Challenges in approximating Koopman eigenvalues and eigenfunctions from data include:

- V is an **unbounded** operator, with **non-isolated eigenvalues** and a non-trivial **continuous spectrum**
- A basis for H is not known a priori

We approach this problem by constructing a data-driven **kernel integral operator** K that commutes with U^t (G. et al 2015; G. 2017; Das & G. 2017)

- Commuting operators have **common eigenspaces**
- The eigenspaces of K are **finite-dimensional**
- In a basis of H consisting of eigenfunctions of K, the eigenvalue problem for f V reduces to that of a skew-symmetric matrix



$F_Q(x_n)$
$F_Q(x_0)$
$\mathbb{R}^{qd}$
F_Q
obs.map in
delay space
$F(x_n)$
$F(x_0)$
$\mathbb{R}^d$
$x_n = \Phi^{n\tau} x_0$
x_0
X
F
obs. map
f
observable
$f_i = f(a_i)$
$\mathbb{R}$

Kernels from delay-embedded data (G. & Majda 2012; Berry et al. 2013; G. 2017; Das & G. 2017)

- Start from the **pseudometric function** $d_Q : X \times X \mapsto \mathbb{R}$,

$$d_Q^2(x, x') = \frac{1}{Q} \sum_{q=0}^{Q-1} \| F(\Phi^{-q\tau}(x)) - F(\Phi^{-q\tau}(x')) \|^2$$

- Choose a bounded shape function $h : \mathbb{R}_+ \mapsto \mathbb{R}$, and define $k_Q : X \times X \mapsto \mathbb{R}$; e.g.,

$$k_Q(x, x') = \exp\left(-\frac{d_Q^2(x, x')}{\epsilon} \right)$$

- Construct the self-adjoint, kernel integral operator $K_Q : H \mapsto H$,

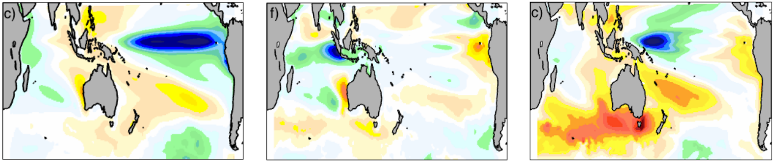
$$K_Q f = \int_X k_Q(\cdot, x) f(x) \, d\mu(x),$$

  and compute the eigenfunctions $K_Q \varphi_j = \lambda_j \varphi_j$

- **Key property:** As $Q \to \infty$, K_Q converges in operator norm to a compact operator K_∞ that commutes with U^t:

$$[U^t, K_\infty] = U^t K_\infty - K_\infty U^t = 0$$

Indo-Pacific variability on seasonal to decadal timescales

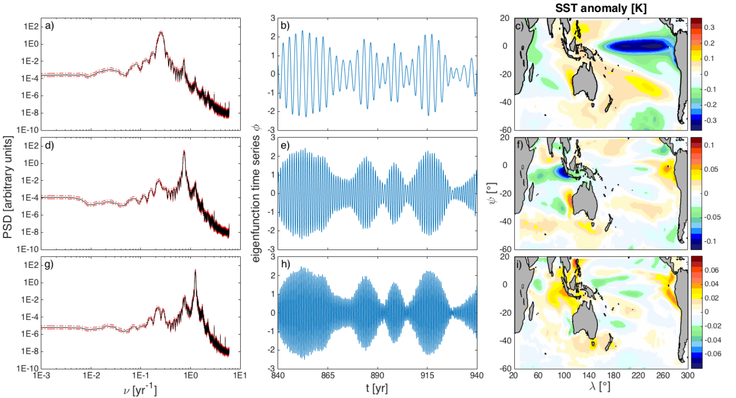


Indo-Pacific sea surface temperature (SST) modes recovered from model (CCSM4, GFDL CM3) and observational (HadISST, 20CRv2) data (Slawinska & G. 2017; G. & Slawinska 2017):

1. Annual cycle and its harmonics
2. ENSO and ENSO combination modes (McGregor et al. 2012; Stuecker et al. 2013)
3. Tropospheric biennial oscillation (TBO) (Meehl 1987) and associated combination modes
4. Interdecadal Pacific oscillation (IPO) (Power et al. 1999)
5. West Pacific multidecadal mode (WPMM)

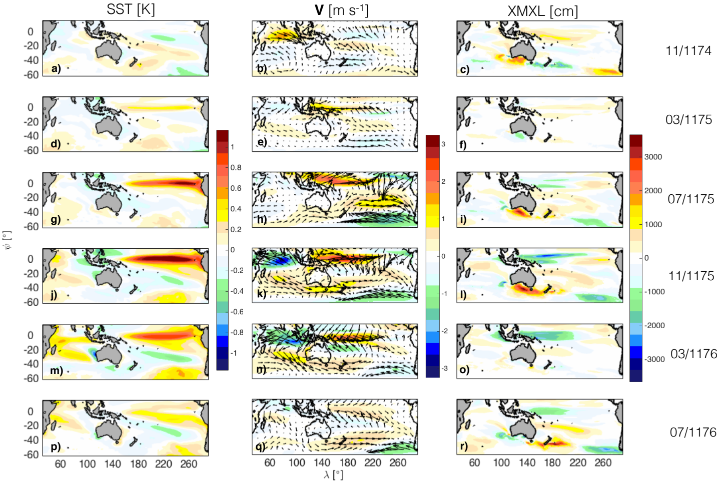
- No bandpass filtering or detrending was necessary to recover this hierarchy of modes
- Additional trend modes representing climate change are captured (ongoing work)

ENSO and ENSO combination modes



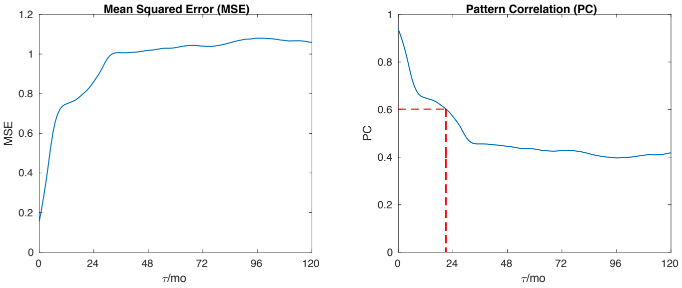
- ENSO emerges as an oscillatory pair of eigenfunctions with a $\nu_{\text{ENSO}} \approx 4\ \text{y}^{-1}$ frequency peak and a **decadal amplitude envelope**
- **Combination modes** predicted from quadratic nonlinearities in the coupled atmosphere-ocean system (McGregor et al. 2012; Stuecker et al. 2013) are recovered at the theoretically expected frequencies $\nu_{\text{ENSO}} \pm 1\ \text{y}^{-1}$ and degeneracies
- The ENSO and ENSO combination modes together capture the phase-locking of ENSO to the annual cycle

ENSO and ENSO combination modes



- Reconstructed surface winds exhibit **anomalous westerlies** during the development of El Niño events in boreal winter, and a **southward shift** preceding El Niño decay in boreal spring (Vecchi 2006; Stuecker et al. 2013)
- Surface circulation consistent with **Indian Ocean SST dipole** (Saji et al. 1999; Webster et al. 1999)

Nonparametric ENSO forecasts

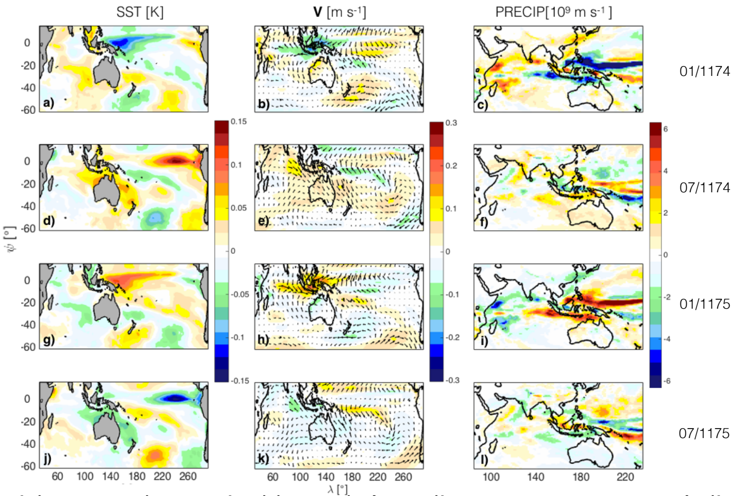


- Using **out-of-sample extension** techniques (Coifman & Lafon 2006) we estimate the **conditional expectation** $\mathbb{E}(U^t f \mid F)$,

$$f = \text{predictand (Niño 3.4)}, \quad F = \text{observation map (Indo-Pacific SST)}$$

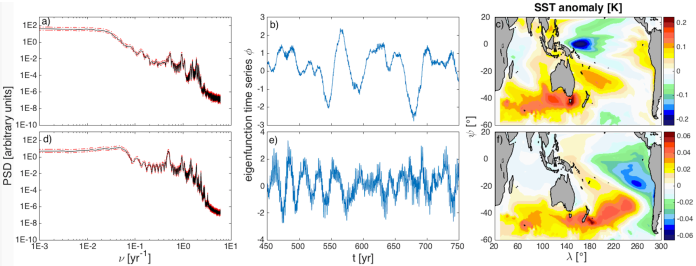
- Approach can be thought of as a generalization of **analog prediction** (Lorenz 69)

Tropospheric biennial oscillation

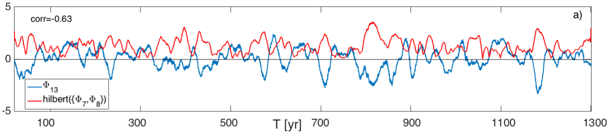


- Biennial pattern characterized by weak Australian monsoon → strong Indian monsoon → strong Australian monsoon → weak Indian monsoon → weak Australian monsoon → . . . (Meehl 1987–1997; Li et al. 2006)
- Precipitation anomalies consistent with Walker cell circulation anomalies (Meehl & Arblaster 2002)

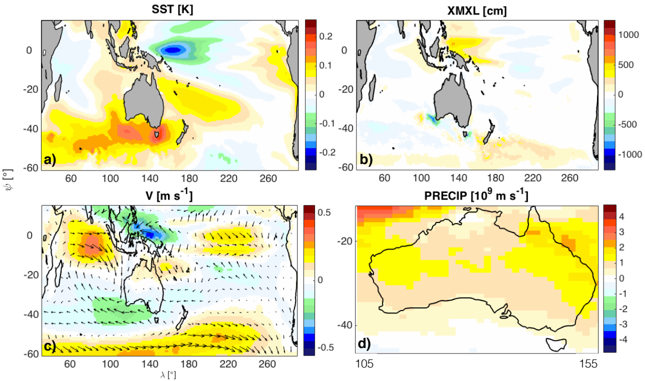
Decadal modes (WPMM & IPO)



- West Pacific multidecadal mode (top) characterized by multidecadal variability and a prominent cluster of SST anomalies in the **western equatorial Pacific**
- Some similarities with 2nd EOF of decadal Pacific SST (Timmermann 2003; Ogata et al. 2013) and ENSO residuals (Karnauskas et al. 2009; Solomon & Newman 2012; Seager et al. 2015)
- Cold (warm) WPMM phases correlate with enhanced (suppressed) ENSO activity (corr. coeff. 0.63 in CCSM4)

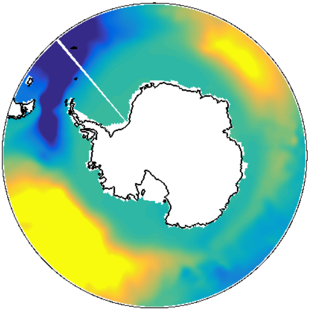


West Pacific multidecadal mode – climate impacts



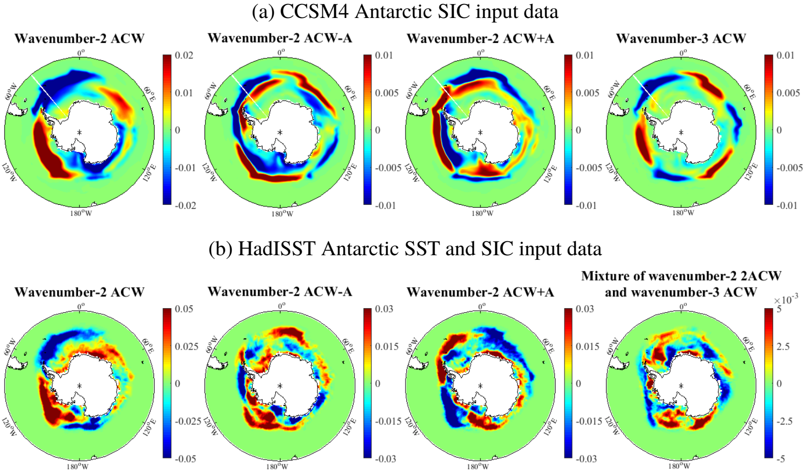
- Cold WPMM phase is characterized by **anomalous westerlies in the central Pacific** and **anomalously flat zonal thermocline profile**; such conditions are known to correlate with enhanced ENSO activity (Kirtman & Schopf 1998; Kleeman et al. 1999; Fedorov & Philander 2000)
- Circulation and SST patterns are consistent with strong impacts on Australian decadal precipitation (corr. coeff. ∼ 0.6)

Antarctic circumpolar waves



- Antarctic SST and sea ice variability on interannual to decadal timescales features prominent traveling and standing modes dominated by wavenumber 2 and 3 patterns (White et al. 1996; Yuan & Martinson 2001; Venegas 2003; Cerrone et al. 2017)
- Wavenumber 2 modes exhibit strong ENSO teleconnections through the PSA pattern (Cai & Baines 2001)

ACWs recovered from CCSM4 and HadISST data (Wang et al. 2017)



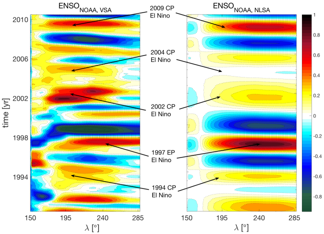
- Leading interannual mode recovered from Antarctic SST data captures the wavenumber 2 ACW, and correlates strongly (PC $\simeq 0.9$ in CCSM4) with ENSO mode from Indo-Pacific
- Associated combination modes represent counter-propagating circumpolar waves

Summary

The **spectral theory of dynamical systems** combined with **kernel methods** from machine learning is a useful approach for characterizing variability of the climate system

- By targeting intrinsic operators to the dynamical system generating, the recovered patterns have high **physical interpretability** while avoiding the need for data prefiltering

Related approaches with applications to climate dynamics include:



- Spectral analysis of **vector-valued observables** for capturing **ENSO diversity** (G. et al. 2017a,b; Slawinska et al. 2018)
- **Analog forecasting approaches** and nonparametric estimation of conditional expectation (Zhao & G. 2016; Alexander et al. 2017)
- Spectral analysis of **skew-product dynamical systems** for detecting coherent patterns of Lagrangian tracers (G. & Das 2017)

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